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# Gauss-Bonnet Formula

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Heather Ann Broersma

December 2006

# Gauss-Bonnet Formula

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December 2006

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## Abstract

Classical differential geometry is the study of curves and surfaces in Euclidean 3-space. From fundamental forms to curvatures, and geodesics, differential geometry has many special theorems and applications worth examining. Among these, the Gauss-Bonnet Theorem is one of the well-known theorems in classical differential geometry. It links geometrical and topological properties of a surface.

In this project, I will:

1. introduce some basic concepts in differential geometry and explain them through some examples

2. extensively analyze and explain the Gauss-Bonnet theorem, then present the proof of the theorem in detail with my own understanding.

3. and, consider applications of the Gauss-Bonnet theorem to some special surfaces.

## Acknowledgements

A special thanks to all the instruction, direction, guidance, assistance, corrections, patience, and goodness of Dr. Wang from the first day of Math 610 (Differential Geometry) until the conclusion of Math 600 (Thesis and Presentation). Also, I would like to thank Dr. Sarli and Dr. Stanton for their valuable comments and great assistance.

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# Chapter 1

# Introduction to Differential Geometry

In this chapter, we will introduce some essential notations and definitions of differential geometry that set the foundation for my study of the Gauss-Bonnet theorem. The exact definitions have been studied and presented from Chuan-Chih Hsiung's book titled, "A First Course in Differential Geometry," reference [Hsi97].

## 1.1 Surface

**Definition.** A **surface** in $E^3$ is a subset S of $E^3$ such that for each point $\mathbf{p} \in S$ there are a neighborhood $V$ of $\mathbf{p}$ in $E^3$ and a mapping $X : U \longrightarrow V \cap E^2$ onto $V \cap S \subset E^3$ subject to the following three conditions:

(i) $X$ is continuously differentiable in $E^3$,

(ii) $X$ is a homeomorphism,

(iii) $X$ is regular at each point $\mathbf{q} \in U$.

The mapping $X$ is called a *parametrization* or a *(local) coordinate system* at $\mathbf{p}$. Condition (ii): $X$ is a homeomorphism implies that $X$ has a continuous inverse $X^{-1}$. Condition (iii) means that for each $\mathbf{q} \in U$, the differential mapping $dX : E^2 \longrightarrow E^3$ is injective. This also implies that at each $\mathbf{q} \in U$ the vector product

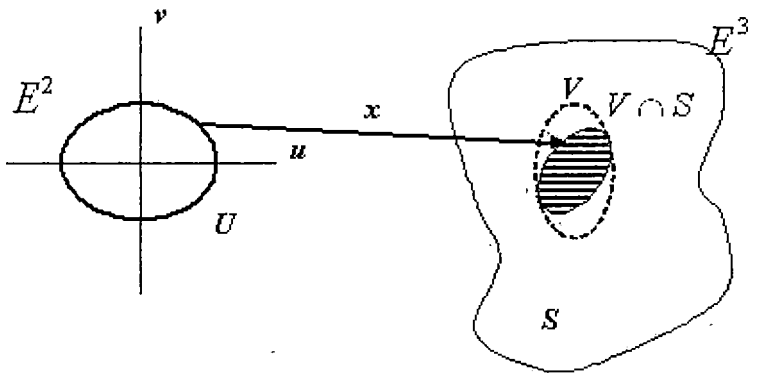


Figure 1.1: The parametrization of a surface.

$$\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \neq 0,$$

where $(u, v) \in U$ (i.e. the vectors $\frac{\partial X}{\partial u}$ and $\frac{\partial X}{\partial v}$ are linearly independent at each $\mathbf{q}$). We will denote these two vectors as

$$X_u = \frac{\partial X}{\partial u}, \qquad X_v = \frac{\partial X}{\partial v}.$$

**Definition.** A **curve** $\alpha : I \longrightarrow S$ on a surface $S$ is a differentiable function from an interval $I$ into $S$, where $I = [a, b]$ on the real line $E^1$.

**Definition.** Let $\mathbf{p}$ be a point of the surface $S$ in $E^3$. A vector in $E^3$ is a **tangent vector** of $S$ at $\mathbf{p}$ if it is tangent to some curve on $S$ at $\mathbf{p}$. A **tangent vector field** on a surface $S$ or on some region $R$ of $S$ is a function that assigns to each point $\mathbf{p}$ of $S$ or $R$ a tangent vector of $S$ at $\mathbf{p}$.

**Lemma.** All tangent vectors of a surface $S$ at a point $\mathbf{p}$ form a plane that is called the tangent plane of $S$ at $\mathbf{p}$ and is denoted by $T_p(s)$.

The tangent plane of a surface through a point $\mathbf{p}$ is the set of all vectors tangent to some curve on the surface passing through $\mathbf{p}$. The tangent plane can be described by the set:

$$T_p = \{Y : \mathbf{p} + \mathbf{s}X_u + \mathbf{t}X_v\} \text{ where } (Y - \mathbf{p}) \cdot (X_u \times X_v) = 0.$$

Note: $X_u$ and $X_v$ are linear independent.

**Definition.** The line orthogonal to the tangent plane $T_p(s)$ of a surface $S$ at a point $\mathbf{p}$ is the **normal** to $S$ at $\mathbf{p}$. A **normal vector field** on $S$ or on a region $R$ of $S$ is a function that assigns to each point $\mathbf{p}$ of $S$ of $R$ a normal vector of $S$ at $\mathbf{p}$.

The unit normal vector to the tangent plane, denoted by $e_3$, is defined as:

$$e_3 = \pm \frac{X_u \times X_v}{| X_u \times X_v |}.$$

The normal plane for a curve through a point $t$ can be described by the equation:

$$(Y - X(t)) \cdot X'(t) = 0,$$

where $X(t)$ is the parametrization of the curve and $X'(t)$ is the tangent to the curve at $t$.

## 1.2 Fundamental Forms of a Surface

Before we state what the fundamental form is, let us look at the differential of a parametrization of surface and integrate its arc length.

Let $X : U \subset E^2 \longrightarrow S$ be a parametrization of a surface $S$, and let $(u, v) \in U$. At a point $\mathbf{x} = X(u, v)$ on $S$, define two tangent vectors $e_1$, $e_2$ such that:

*(i)* $e_1(u, v)$ and $e_2(u, v)$ are of continuously differentiable in $E^2$; and

*(ii)* the determinant $|e_1, e_2, e_3| > 0$.

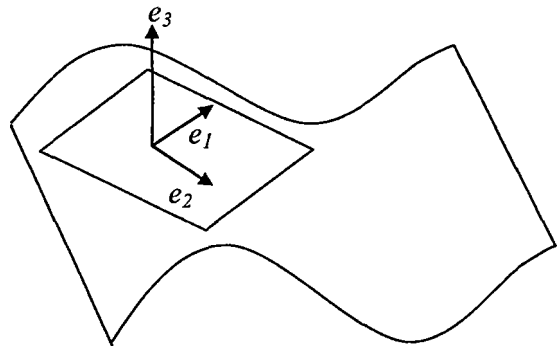


Figure 1.2: The tangent vectors at a point on a surface.

For a $X(u,v)$, we have $X_u$ and $X_v$, which can be expressed as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. We have:

$$X_u = p_1\mathbf{e}_1 + p_2\mathbf{e}_2,$$

$$X_v = q_1\mathbf{e}_1 + q_2\mathbf{e}_2,$$

where $p_1$, $p_2$, $q_1$, and $q_2$ are function of $u$ and $v$. Now we will look at the differential of $X$ to construct the inner product in order to define the first fundamental form of the surface at a point $\mathbf{x}$,

$$dX = X_u du + X_v dv.$$

THE FIRST FUNDAMENTAL FORM:

$$\mathbf{I} = dX \cdot dX. \tag{1.1}$$

Take:

$$\begin{aligned} dX \cdot dX &= (X_u du + X_v dv) \cdot (X_u du + X_v dv) \\ &= (X_u \cdot X_u)du^2 + (X_u \cdot X_v)dudv + (X_v \cdot X_u)dvdu + (X_v \cdot X_v)dv^2 \\ &= (X_u \cdot X_u)du^2 + 2(X_u \cdot X_v)dudv + (X_v \cdot X_v)dv^2 \\ &= Edu^2 + 2Fdudv + Gdv^2, \end{aligned}$$

where

$$E = X_u \cdot X_u, \qquad F = X_u \cdot X_v, \qquad G = X_v \cdot X_v.$$

The arc length of $X(t)$ in a surface, from $t_1$ to $t_2$ is given by

$$s = \int_{t_1}^{t_2} \|X'(t)\| dt.$$

Therefore, the first fundamental form can be written as follows:

$$\mathbf{I} = Edu^2 + 2Fdudv + Gdv^2. \tag{1.2}$$

**Definition.** An **isometry** $f : S \longrightarrow \bar{S}$ of two surfaces $S, \bar{S}$ in $E^3$ is a bijective differentiable mapping that preserves the first fundamental form; when this is the case, the two surfaces $S, \bar{S}$ are said to be **isometric.**

**Example.** A plane, parameterized by

$$x(u, v) = (0, u, v),$$

and a cylinder, parameterized by

$$x(u, v) = (\cos u, \sin u, v), \qquad \text{such that } 0 < u < 2\pi,$$

are isometric since they both have the first fundamental form:

$$\mathbf{I} = du^2 + dv^2.$$

Now we will look at the differential of $e_3$ to construct the inner product in order to define the second fundamental form of the surface at a point $\mathbf{x}$,

$$d\mathbf{e}_3 = \mathbf{e}_{3u}du + \mathbf{e}_{3v}dv.$$

THE SECOND FUNDAMENTAL FORM:

$$\mathbf{II} = -dX \cdot de_3. \tag{1.3}$$

Take:

$$\begin{aligned} -dX \cdot de_3 &= (X_u du + X_v dv) \cdot (e_{3u}du + e_{3v}dv) \\ &= -\left((X_u \cdot e_{3u})du^2 + (X_u \cdot e_{3v})dudv + (X_v \cdot e_{3u})dvdu + (X_v \cdot e_{3v})dv^2\right) \\ &= -\left((e_3 \cdot X_{uu})du^2 + 2(e_3 \cdot X_{uv})dudv + (e_3 \cdot X_{vv})dv^2\right) \\ &= Ldu^2 + 2MFdudv + Ndv^2, \end{aligned}$$

where

$$L = -e_{3u} \cdot X_u = e_3 \cdot X_{uu},$$

$$M = -e_{3u} \cdot X_v = -e_{3v} \cdot X_u = e_3 \cdot X_{uv},$$

$$N = -e_{3v} \cdot X_v = e_3 \cdot X_{vv}.$$

Therefore, the second fundamental form can be written as follows:

$$\mathbf{II} = Ldu^2 + 2Mdudv + Ndv^2. \tag{1.4}$$

THE THIRD FUNDAMENTAL FORM:

$$\mathbf{III} = de_3 \cdot de_3. \tag{1.5}$$

The fundamental forms **I**, **II**, **III** can be regarded as a function on $du$ and $dv$.

## 1.3 Curvature

We will now look at the relationship among the curvatures of curves on a surface, $S$ at a point $\mathbf{p}$.

Let $X : U \subset E^2 \longrightarrow S$ be a parametrization of a surface $S$, and let $(u, v) \in U$. Let $C$ be a curve of $S$ through $\mathbf{p} = X(u, v)$ with arc length $s$, and let $\mathbf{t}$ be the unit tangent vector and $\mathbf{n}$ be the principal normal vector of $C$ at $\mathbf{p}$. The curvature $\kappa$ of $C$ at $\mathbf{p}$ is then given by:

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n}.$$

**Definition.** The **normal curvature**, denoted by $\kappa_N$ is the curvature of the curve from the intersection of a surface by a plane through the unit tangent vector, $\mathbf{t}$ and surface normal, $e_3$.

If $\theta$ denotes the angle between the principal normal vector $\mathbf{n}$ and the surface normal $e_3$, we have the following relation:

$$\kappa_N = \kappa \cos\theta.$$

The normal curvature can also be written as the quotient of the second and first fundamental form:

$$\begin{aligned}
\kappa \cos\theta &= \kappa(n \cdot e_3) \\
&= \frac{dt}{ds} \cdot e_3 \\
&= -t \cdot \frac{de_3}{ds} \\
&= -\frac{dx}{ds} \cdot \frac{de_3}{ds} \\
&= \frac{-dx \cdot de_3}{ds \cdot ds} \\
&= \frac{\mathbf{II}}{\mathbf{I}}.
\end{aligned}$$

We will be expressing the normal curvature using the following:

$$\kappa_N = \frac{\mathbf{II}}{\mathbf{I}},$$

$$\kappa_N = e_2 \cdot e_3,$$

$$\kappa_N = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

**Example.** If we have a unit speed curve $\gamma(t)$ on $X$ then $\kappa_N = \mathbf{II}$, since:

$$\begin{aligned}
\kappa_N &= e_3 \cdot \ddot{X} \\
&= e_3 \cdot \frac{d}{dt}\dot{X} \\
&= e_3 \cdot \frac{d}{dt}(X_u\dot{u} + X_v\dot{v}) \\
&= e_3 \cdot (X_u\ddot{u} + X_v\ddot{v} + (X_{uu}\dot{u} + X_{uv}\dot{v})\dot{u} + (X_{uv}\dot{u} + X_{vv}\dot{v})\dot{v}) \\
&= Ldu^2 + 2Mdudv + Ndv^2.
\end{aligned}$$

**Definition.** The **principal curvatures** of a surface patch are the roots $k_1, k_2$ of the equation

$$\begin{vmatrix} L - kE & M - kF \\ M - kF & N - kG \end{vmatrix} = 0.$$

**Definition.** From the principal curvatures we have, the **mean curvature** defined as:

$$H = \frac{1}{2}(k_1 + k_2)$$

or

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

Note: A surface with $H = 0$ everywhere is called a minimal surface.

**Definition.** Also, from the principal curvatures we have the **Gaussian curvature** defined as:

$$K = k_1 k_2.$$

We can express the Gaussian curvature with the fundamental form coefficients:

$$K = \frac{LN - M^2}{EG - F^2}.$$

By looking at the Gaussian curvature, we can determine if a point on the space is elliptic, hyperbolic, parabolic, or planar. The following surfaces have constant Gaussian curvature. In the chart below the type of a surface and the type of a point on a surface is known by calculating the Gaussian curvature.

| Gaussian curvature | Type of point on a surface | Example of a surface |
|---|---|---|
| $K = 0$ | parabolic or planer point | plane |
| $K > 0$ (constant) | elliptic point | sphere |
| $K < 0$ (constant) | hyperbolic point | pseudosphere |

There is no complete surface in Euclidean 3-space of constant negative Gaussian curvature. However, if a compact connected surface has constant Gaussian curvature $K$

then the surface is a sphere of radius:

$$\frac{1}{\sqrt{K}}.$$

The Gaussian curvature is also preserved by isometries. If we have some isometry $f : S_1 \to S_2$ then the Gaussian curvature of $S_1$ at a point $p$ is equal to the Gaussian curvature of $S_2$ at $f(p)$.

By calculating the mean and Gaussian curvature by means of the first and second fundamental forms of a surface, we can, in turn, find the principal curvature $k_1$, $k_2$:

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

## 1.4 Christoffel Symbols

Is there a link between the second fundamental form and the first fundamental form? The Christoffel symbols express $X_{uu}$, $X_{uv}$, $X_{vv}$, $\mathbf{e}_{3u}$, $\mathbf{e}_{3v}$ as a basis of $X_u$, $X_v$, $\mathbf{e}_3$ where $X : (u, v) \to S$ is a parametrization of a surface $S$, and $\mathbf{e}_3$ is the unit normal vector of $S$. The formulas for $X_{uu}$, $X_{uv}$, and $X_{vv}$ are:

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L\mathbf{e}_3, \tag{1.6}$$

$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M\mathbf{e}_3, \tag{1.7}$$

$$X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N\mathbf{e}_3, \tag{1.8}$$

where $\Gamma^i_{jk}$ for $i, j, k = 1, 2$ are called the Christoffel symbols. The Christoffel symbols are to be determined as follows.

We will express $\mathbf{e}_{3u}$, $\mathbf{e}_{3v}$ as a basis of $X_u, X_v$. The partial derivatives can be expressed as

$$\begin{aligned} \mathbf{e}_{3u} &= aX_u + bX_v, \\ \mathbf{e}_{3v} &= cX_u + dX_v. \end{aligned}$$

By taking the inner product of these with $X_u$, $X_v$,

$$\text{i.e. } e_{3u} \cdot X_u \ , \ e_{3u} \cdot X_v \ , \ e_{3v} \cdot X_u \ , \ e_{3v} \cdot X_v$$

we have:

$$\begin{cases} -L &= aE + bF \\ -M &= aF + bG \end{cases}, \tag{1.9}$$

$$\begin{cases} -M &= cE + dF \\ -N &= cF + dG \end{cases}, \tag{1.10}$$

for some coefficients $a, b, c, d$. Recall from the fundamental forms, the following dot products with $e_3$:

$$\begin{aligned} e_3 \cdot X_{uu} &= -e_{3u} \cdot X_u = L, \\ e_3 \cdot X_{uv} &= -e_{3v} \cdot X_u = -e_{3u} \cdot X_v = M, \\ \text{and} \quad e_3 \cdot X_{vv} &= -e_{3v} \cdot X_v = N. \end{aligned}$$

Solving the system (1.9) and (1.10) for $a, b, c, d$ gives

$$a = \frac{-LG + FM}{EG - F^2}, \qquad b = \frac{-EM + LF}{EG - F^2},$$

$$c = \frac{-MG + NF}{EG - F^2}, \qquad d = \frac{-EN + FM}{EG - F^2}.$$

Therefore we have $\mathbf{e}_{3u}$ and $\mathbf{e}_{3v}$ written as linear independent combinations of $X_u$ and $X_v$:

$$\begin{aligned} e_{3u} &= \frac{(FM - LG)X_u + (FL - EM)X_v}{EG - F^2}, \\ e_{3v} &= \frac{(FN - GM)X_u + (FM - EN)X_v}{EG - F^2}. \end{aligned} \tag{1.11}$$

These are also known as the Weingarten formulas.

Now, we will express $X_{uu}$, $X_{uv}$, $X_{vv}$ as a basis of $\{X_u, X_v, e_3\}$. Consider the derivatives of $X_u$, $X_v$:

$$dX_u = X_{uu} + X_{uv},$$

$$dX_v = X_{uv} + X_{vv}.$$

The partial derivatives can be expressed as

$$X_{uu} = \alpha_1 X_u + \alpha_2 X_v + \alpha_3 \mathbf{e}_3, \tag{1.12}$$

$$X_{uv} = \beta_1 X_u + \beta_2 X_v + \beta_3 \mathbf{e}_3, \tag{1.13}$$

$$X_{vv} = \gamma_1 X_u + \gamma_2 X_v + \gamma_3 \mathbf{e}_3. \tag{1.14}$$

By taking the inner product of the partial derivatives with $e_3$, we have:

$$\begin{aligned} X_{uu} \cdot \mathbf{e}_3 &= \alpha_1 X_u \cdot \mathbf{e}_3 + \alpha_2 X_v \cdot \mathbf{e}_3 + \alpha_3 \mathbf{e}_3 \cdot \mathbf{e}_3 = L, \\ X_{uv} \cdot \mathbf{e}_3 &= \beta_1 X_u \cdot \mathbf{e}_3 + \beta_2 X_v \cdot \mathbf{e}_3 + \beta_3 \mathbf{e}_3 \cdot \mathbf{e}_3 = M, \\ X_{vv} \cdot \mathbf{e}_3 &= \gamma_1 X_u \cdot \mathbf{e}_3 + \gamma_2 X_v \cdot \mathbf{e}_3 + \gamma_3 \mathbf{e}_3 \cdot \mathbf{e}_3 = N, \end{aligned}$$

which becomes

$$L = \alpha_3, \qquad M = \beta_3, \qquad N = \gamma_3.$$

By taking the inner product of the partial derivative $X_{uu}$ with $X_u$, $X_v$, we have:

$$X_{uu} \cdot X_u = \alpha_1 X_u \cdot X_u + \alpha_2 X_v \cdot X_u + \alpha_3 \mathbf{e}_3 \cdot X_u,$$

$$X_{uu} \cdot X_v = \alpha_1 X_u \cdot X_v + \alpha_2 X_v \cdot X_v + \alpha_3 \mathbf{e}_3 \cdot X_v.$$

Note: for any parameterization $R$, the following holds true

$$R_{uu} \cdot R_u = \frac{1}{2}(R_u \cdot R_u)_u \text{ and } R_{uu} \cdot R_v = (R_u \cdot R_v)_u - R_u \cdot R_{uv}.$$

Therefore, if we solve this system of equations,

$$\begin{cases} X_{uu} \cdot X_u = \alpha_1 E + \alpha_2 F \\ X_{uu} \cdot X_v = \alpha_1 F + \alpha_2 G \end{cases},$$

where

$$\begin{cases} X_{uu} \cdot X_u = (X_u \cdot X_u)_u = \frac{1}{2} E_u \\ X_{uu} \cdot X_v = (X_u \cdot X_v)_u - X_u \cdot X_{uv} = F_u - \frac{1}{2} E_v \end{cases},$$

we can obtain

$$\begin{aligned} \alpha_1 &= \frac{\begin{vmatrix} \frac{1}{2} E_u & F \\ F_u - \frac{1}{2} E_v & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} \\ &= \frac{\frac{1}{2} E_u G - F F_u + \frac{1}{2} E_v F}{EG - F^2} \\ &= \frac{G E_u - 2 F F_u + F E_v}{2(EG - F^2)} \\ &= \Gamma_{11}^1 \end{aligned}$$

and

$$\alpha_2 = \frac{\begin{vmatrix} E & \frac{1}{2}E_u \\ F & F_u - \frac{1}{2}E_v \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$$= \frac{EF_u - \frac{1}{2}EE_v - \frac{1}{2}FE_u}{EG - F^2}$$

$$= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}$$

$$= \Gamma_{11}^2.$$

By taking the inner product of the partial derivative $X_{uv}$ with $X_u$, $X_v$, we have:

$$X_{uv} \cdot X_u = \beta_1 X_u \cdot X_u + \beta_2 X_v \cdot X_u + \beta_3 \mathbf{e}_3 \cdot X_u,$$

$$X_{uv} \cdot X_v = \beta_1 X_u \cdot X_v + \beta_2 X_v \cdot X_v + \beta_3 \mathbf{e}_3 \cdot X_v.$$

If we solve this system of equations,

$$\begin{cases} X_{uv} \cdot X_u = \beta_1 E + \beta_2 F \\ X_{uv} \cdot X_v = \beta_1 F + \beta_2 G \end{cases},$$

where

$$\begin{cases} X_{uv} \cdot X_u = \frac{1}{2}E_v \\ X_{uv} \cdot X_v = \frac{1}{2}G_u \end{cases},$$

we can obtain

$$\beta_1 = \frac{\begin{vmatrix} \frac{1}{2}E_v & F \\ \frac{1}{2}G_u & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$$= \frac{\frac{1}{2}E_vG - \frac{1}{2}FG_u}{EG - F^2}$$

$$= \frac{GE_v - FG_u}{2(EG - F^2)}$$

$$= \Gamma_{12}^1$$

and

$$\beta_2 = \frac{\begin{vmatrix} E & \frac{1}{2}E_v \\ F & \frac{1}{2}G_u \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$$= \frac{\frac{1}{2}EG_u - \frac{1}{2}FE_v}{EG - F^2}$$

$$= \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$= \Gamma_{12}^2.$$

Finally, by taking the inner product of the partial derivative $X_{uv}$ with $X_u$, $X_v$, we have:

$$X_{vv} \cdot X_u = \gamma_1 X_u \cdot X_u + \gamma_2 X_v \cdot X_u + \gamma_3 \mathbf{e}_3 \cdot X_u,$$

$$X_{vv} \cdot X_v = \gamma_1 X_u \cdot X_v + \gamma_2 X_v \cdot X_v + \gamma_3 \mathbf{e}_3 \cdot X_v.$$

If we solve this system of equations,

$$\begin{cases} X_{vv} \cdot X_u = \gamma_1 E + \gamma_2 F \\ X_{vv} \cdot X_v = \gamma_1 F + \gamma_2 G \end{cases},$$

where

$$\begin{cases} X_{vv} \cdot X_u = (X_v \cdot X_u)_v - X_v \cdot X_{vu} = F_v - \frac{1}{2}G_u \\ X_{vv} \cdot X_v = (X_v \cdot X_v)_v = G_u \end{cases},$$

we can obtain

$$\gamma_1 = \frac{\begin{vmatrix} F_v - \frac{1}{2}G_u & F \\ \frac{1}{2}G_v & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$$= \frac{GG_u - \frac{1}{2}GF_v - \frac{1}{2}FG_v}{EG - F^2}$$

$$= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$= \Gamma_{22}^1$$

and

$$\gamma_2 = \frac{\begin{vmatrix} E & F_v - \frac{1}{2}G_u \\ F & \frac{1}{2}G_v \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$$= \frac{\frac{1}{2}EG_v - FF_v + \frac{1}{2}FG_u}{EG - F^2}$$

$$= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

$$= \Gamma_{22}^2.$$

Therefore, we have the Christoffel symbols which only depend on the first fundamental form.

Now we will consider higher partial derivatives of $X$ and $e_3$ by substituting (1.6) and (1.7) into the equation

$$(X_{uu})_v = (X_{uv})_u.$$

The left hand side simplifies to the following using (1.8) and (1.11):

$$\begin{aligned}
(X_{uu})_v &= (\Gamma^1_{11}X_u + \Gamma^2_{11}X_v + Le_3)_v \\
&= (\Gamma^1_{11})_v X_u + \Gamma^1_{11}X_{uv} + (\Gamma^2_{11})_v X_v + \Gamma^2_{11}X_{vv} + L_v e_3 + Le_{3v} \\
&= (\Gamma^1_{11})_v X_u + \Gamma^1_{11}(\Gamma^1_{12}X_u + \Gamma^2_{12}X_v + Me_3) + (\Gamma^2_{11})_v X_v \\
&\quad + \Gamma^2_{11}(\Gamma^1_{22}X_u + \Gamma^2_{22}X_v + Ne_3) + L_v e_3 \\
&\quad + L\left(\frac{(FN-GM)X_u + (FM-EN)X_v}{EG-F^2}\right) \\
&= \left[(\Gamma^1_{11})_v + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + L\frac{FN-GM}{EG-F^2}\right] X_u \\
&\quad + \left[(\Gamma^2_{11})_v + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + L\frac{FM-EN}{EG-F^2}\right] X_v \\
&\quad + \left[L_v + M\Gamma^1_{11} + N\Gamma^2_{11}\right] e_3. \qquad (1.15)
\end{aligned}$$

The right hand side simplifies to the following:

$$\begin{aligned}
(X_{uv})_u &= (\Gamma^1_{12}X_u + \Gamma^2_{12}X_v + Me_3)_u \\
&= (\Gamma^1_{12})_u X_u + \Gamma^1_{12}X_{uu} + (\Gamma^2_{12})_u X_v + \Gamma^2_{12}X_{vu} + M_u e_3 + Me_{3u} \\
&= (\Gamma^1_{12})_u X_u + \Gamma^1_{12}(\Gamma^1_{11}X_u + \Gamma^2_{11}X_v + Le_3) + (\Gamma^2_{12})_u X_v \\
&\quad + \Gamma^2_{12}(\Gamma^1_{12}X_u + \Gamma^2_{12}X_v + Me_3) + M_u e_3 \\
&\quad + M\left(\frac{(FM-LG)X_u + (FL-EM)X_v}{EG-F^2}\right)
\end{aligned}$$

$$= \left[(\Gamma^1_{12})_u + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + M\frac{FM - LG}{EG - F^2}\right] X_u$$

$$+ \left[(\Gamma^2_{12})_u + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + M\frac{FL - EM}{EG - F^2}\right] X_v$$

$$+ \left[M_u + L\Gamma^1_{12} + M\Gamma^2_{12}\right] e_3. \tag{1.16}$$

And setting the corresponding coefficients of $X_u$, $X_v$, and $e_3$ terms in (1.15) and (1.16) equal to each other, we have:

$$(\Gamma^1_{11})_v + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + L\frac{FN - GM}{EG - F^2} =$$
$$(\Gamma^1_{12})_u + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + M\frac{FM - LG}{EG - F^2},$$

$$\frac{LFN - LGM - FM^2 + LGM}{EG - F^2} =$$
$$(\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^1_{11} - \Gamma^1_{12}\Gamma^1_{11},$$

$$F\frac{LN - M^2}{EG - F^2} = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}; \tag{1.17}$$

$$(\Gamma^2_{11})_v + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + L\frac{FM - EN}{EG - F^2} =$$
$$(\Gamma^2_{12})_u + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + M\frac{FL - EM}{EG - F^2},$$

$$\frac{FML - ENL - FML + EM^2}{EG - F^2} =$$
$$(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22},$$

$$-E\frac{LN - M^2}{EG - F^2} =$$
$$(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}; \tag{1.18}$$

$$L_v + M\Gamma_{11}^1 + N\Gamma_{11}^2 = M_u + L\Gamma_{12}^1 + M\Gamma_{12}^2,$$

$$L_v - M_u = L\Gamma_{12}^1 - N\Gamma_{11}^2 + M(\Gamma_{12}^2 - \Gamma_{11}^1). \tag{1.19}$$

Likewise we can substitute (1.7) and (1.8), into the equation

$$(X_{uv})_v = (X_{vv})_u,$$

and through similar calculations from above (not to be shown here) we have the following results,

$$G\frac{LN - M^2}{EG - F^2} = (\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1\Gamma_{11}^1 - \Gamma_{12}^1\Gamma_{12}^1 + \Gamma_{22}^2\Gamma_{12}^1 - \Gamma_{12}^2\Gamma_{22}^1; \tag{1.20}$$

$$-F\frac{LN - M^2}{EG - F^2} = (\Gamma_{22}^2)_u - (\Gamma_{12}^2)_v + \Gamma_{22}^1\Gamma_{11}^2 - \Gamma_{12}^1\Gamma_{12}^2; \tag{1.21}$$

$$M_v - N_u = L\Gamma_{22}^1 - N\Gamma_{12}^2 + M(\Gamma_{22}^2 - \Gamma_{12}^1). \tag{1.22}$$

The equations (1.17), (1.18), (1.19), (1.20), (1.21), and (1.22) that represent higher partial derivatives of $X$ and $e_3$ are called the **Gauss equations**. And these give us another remarkable concept that the Gaussian curvature only depends on the first fundamental form, namely:

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} - Fuv - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}.$$

If $F = 0$, then the Gaussian curvature simplifies to

$$K = -\frac{1}{2\sqrt{EG}}\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right).$$

And if $E = 1$, we have,

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2} = -\frac{\ddot{f}}{f}.$$

## 1.5 Covariant Differential

Now, we can use the Christoffel symbols to write the differential forms of the parametrization, $X(u, v)$ of a surface, $S$, namely:

$$\begin{aligned} dX_u &= X_{uu}du + X_{uv}dv \\ &= \Gamma^1_{11}X_u du + \Gamma^1_{12}X_u dv + \Gamma^2_{11}X_v du + \Gamma^2_{12}X_v dv + b_{11}e_3 du + b_{12}e_3 dv \end{aligned},$$

$$\begin{aligned} dX_v &= X_{vu}du + X_{vv}dv \\ &= \Gamma^1_{21}X_u du + \Gamma^1_{22}X_u dv + \Gamma^2_{21}X_v du + \Gamma^2_{22}X_v dv + b_{21}e_3 du + b_{22}e_3 dv \end{aligned},$$

where $b_{11} = L, \quad b_{12} = M, \quad and \quad b_{22} = N.$

We previously know how to describe any tangent vector at a point, $p$ on $S$ as a linear combination of $X_u$ and $X_v$.

Let $V$ be the tangent vector at $p$, see Figure (1.3), where

$$V = y_1X_u + y_2X_v,$$

And the differential of $V$ is

$$\begin{aligned} dV &= dy_1X_u + y_1(\Gamma^1_{11}X_u du + \Gamma^1_{12}X_u dv + \Gamma^2_{11}X_v du + \Gamma^2_{12}X_v dv) + y_1(b_{11}e_3 d_u + b_{12}e_3 d_v) \\ &\quad + dy_2X_v + y_2(\Gamma^1_{21}X_u du + \Gamma^1_{22}X_u dv + \Gamma^2_{21}X_v du + \Gamma^2_{22}X_v dv) + y_2(b_{21}e_3 d_u + b_{22}e_3 d_v). \end{aligned}$$

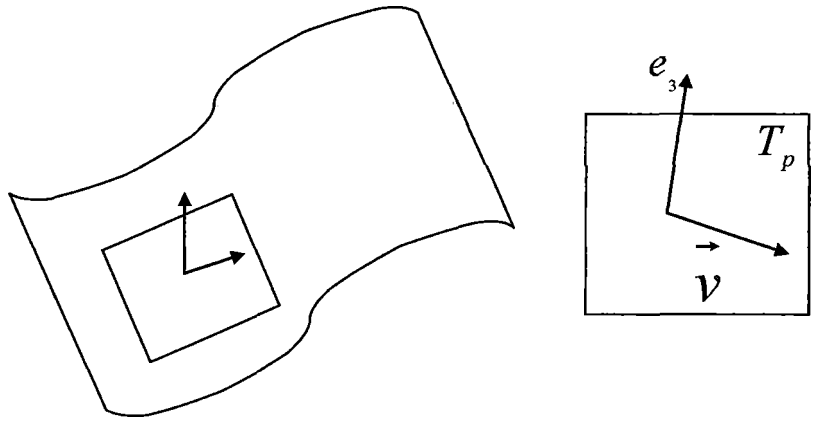


Figure 1.3: The tangent vector $V$.

The change of the vector field in terms of the tangent space can be described by the covariant differential.

**Definition.** The **covariant differential** $\nabla V$ of a tangent vector $V$ of $S$ at a point $p$ is the orthogonal projection of the differential $dV$ from the normal $e_3$ of $S$ at $p$ onto the tangent plane of $S$ at $p$. Therefore,

$$\begin{aligned}\nabla V &= (dy_1 + y_1\Gamma^1_{11}du + y_1\Gamma^1_{12}dv + y_2\Gamma^1_{21}du + y_2\Gamma^1_{22}dv)X_u \\ &\quad +(dy_2 + y_1\Gamma^2_{11}du + y_1\Gamma^2_{12}dv + y_2\Gamma^2_{21}du + y_2\Gamma^2_{22}dv)X_v \\ &= \nabla y_1 X_u + y_1\nabla X_u + \nabla y_2 X_v + y_2\nabla X_v.\end{aligned}$$

Since $e_3$ is orthogonal to $\nabla V$, we have

$$\nabla V \cdot e_3 = 0.$$

Also, the covariant differential $\nabla V$ depends only on the first fundamental form. A special case of the covariant differential is when $\nabla V = 0$.

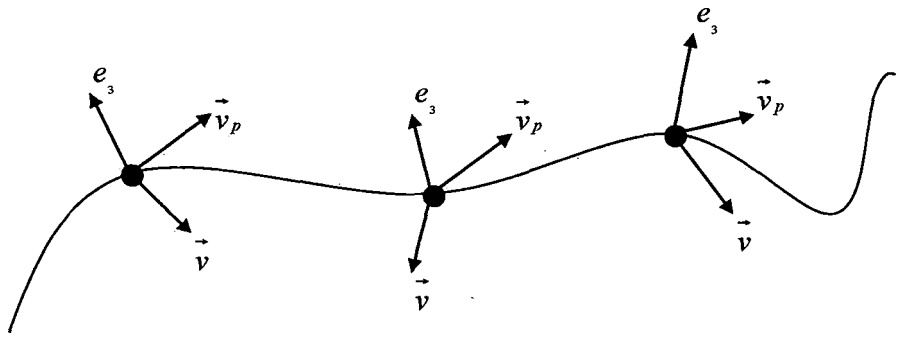


Figure 1.4: The parallel tangent vector field.

**Definition.** The **parallel tangent vector field** of $V$ along a curve $C$ on $S$ is a vector field $V$ satisfying the condition $\nabla V = 0$.

If follows that when $\nabla V = 0$ we also have:

$$\frac{dy_1}{dt} + y_1\Gamma^1_{11}\frac{du}{dt} + y_2\Gamma^1_{22}\frac{dv}{dt} = 0,$$
$$\frac{dy_2}{dt} + y_1\Gamma^2_{11}\frac{du}{dt} + y_2\Gamma^2_{22}\frac{dv}{dt} = 0.$$

In conclusion, we can use parallel tangent vector fields for two important statements.
(1) Any two surfaces with the same first fundamental form, I, have the same parallelism.
(2) The angle between parallel tangent vector fields $V$ and $W$ on $S$ have constant angle $\theta$ if $V$ and $W$ have constant lengths.

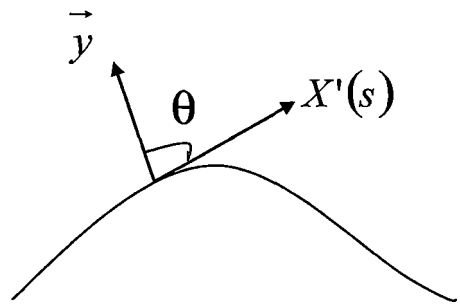


Figure 1.5: The angle between the unit tangent vectors.

## 1.6 Geodesics

It is common knowledge that the shortest distance between two points in a plane is the straight line that joins them. We can generalize the definition of curvature of a plane curve to define the geodesic curvature of a curve $C$ on a surface $S$ by using the notion of parallelism. Recall that the curvature of a plane curve at a point $\mathbf{p}(s)$ can be defined as

$$\kappa = \frac{dt}{ds},$$

where $s$ is the arc length and $t$ is the angle which the oriented tangent of $C$ at $\mathbf{p}$ makes with a fixed direction.

**Definition.** Consider a family of unit tangent vectors $\vec{y}(s)$ of a surface $S$, which are parallel along a curve $C$ on $S$ where $s$ is the arc length of $C$. Let $\theta$ be the angle between $\vec{y}(s)$ and the unit tangent vector $X'(s)$ of $C$. Then the rate of change of $\theta$ with respect to $s$ is called the **geodesic curvature** at a point $\mathbf{p}$ of $C$ on $S$, denoted by

$$\kappa_g = \frac{d\theta}{ds}.$$

The **geodesic** of the surface is a curve on the surface with zero geodesic curvature everywhere.

Therefore, the angle between the unit tangent vector a parallel tangent vector is constant along the geodesic. Note that $\frac{d\theta}{ds}$ is independent of the choice of $\vec{y}$.

Examples of geodesics are: {1} all straight lines in a plane, {2} the images under the isometry of all the straight lines in a plane to a cylinder, {3} all great circles on a sphere and {4} any normal section of a surface.

## 1.7 Euler Characteristic

**Definition.** A **triangulation** of a regular region $R$ of $S$ is a finite family of triangles $\Im_1, ...\Im_n$ such that

(1) the union of all $\Im_i$ covers the region; and

(2) the intersection of any $\Im_i$ with $\Im_j$ is equal to one of the following:

(a) empty,

(b) common side of $\Im_i$ and $\Im_j$,

(c) or common vertex of $\Im_i$ and $\Im_j$.

The triangulation gives us such information as the number of triangles, $f$, the number of edges on the triangles, $e$, and the number of vertices, $v$, on the triangles. It can be proved that $v - e + f$ is independent of triangulations.

**Definition:** The Euler characteristic of the triangulation of regular region of a surface is

$$v - e + f = \chi.$$

Even though different triangulations of a surface, $S$, will have a different number triangles, edges, and vertices, the Euler characteristic only depends on the surface and is independent of triangulations. We will calculate the Euler characteristic of some surfaces in Chapter 3.

## 1.8 Examples

**Example. THE ELLIPSOID**

Now we will look at an example of a surface and calculate the normal, principal, Gaussian, and mean curvature. The ellipsoid is more complex surface to examine than the sphere.

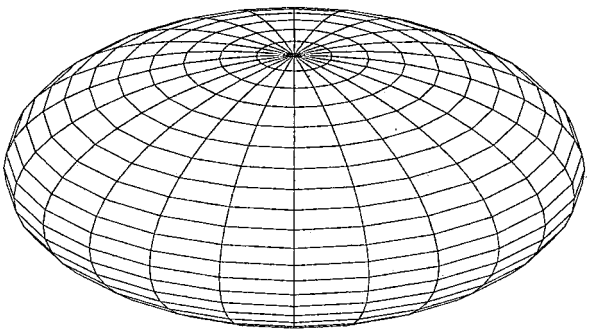

Figure 1.6: The ellipsoid.

The ellipsoid centered at the origin is as follows:

$$S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\},$$

where the $x$,$y$, and $z$ intercepts are $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$.

We can parametrize S by

$$X(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u). \tag{1.23}$$

1. Show the ellipsoid is a surface on the intervals $0 < u < \pi$ and $0 < v < 2\pi$.
   First, to show that $X$ is one to one, we need to show $X(u_1, v_1) = X(u_2, v_2)$. In fact, show

   $$(a \sin u_1 \cos v_1, b \sin u_1 \sin v_1, c \cos u_1) = (a \sin u_2 \cos v_2, b \sin u_2 \sin v_2, c \cos u_2).$$

   Since the function $f(x) = \cos x$ is a one to one function on $0 < x < \pi$, then $u_1 = u_2$ implies that

   $$\cos u_1 = \cos u_2 \quad \text{for any} \quad u_1, \quad u_2 \quad \in (0, \pi).$$

   Therefore,

   $$c \cos u_1 = c \cos u_2 \quad \text{when} \quad u_1 = u_2.$$

   To show that

   $$(a \sin u_1 \cos v_1, b \sin u_1 \sin v_1) = (a \sin u_2 \cos v_2, b \sin u_2 \sin v_2)$$

   we need to consider two cases:

$$(i)\ v_1 = v_2 \qquad \text{and}$$
$$(ii)\ v_1 = 2\pi - v_2.$$

We already have

$$c\cos u_1 = c\cos u_2$$

which immediately gives

$$\sin u_1 = \sin u_2.$$

And $\sin v_1 = \sin v_2$ is obvious when $v_1 = v_2$. If $v_1 = 2\pi - v_2$, then

$$\sin v_1 = \sin(2\pi - v_2) = -\sin v_2.$$

Either

$$\sin v_1 = \sin v_2 = 0 \quad \text{or} \quad v_1 = v_2 = \pi\ ,$$

which both show $\sin v_1 = \sin v_2$. Therefore, we have

$$X(u_1, v_1) = X(u_2, v_2),$$

when $u_1 = u_2$ and $v_1 = v_2$.

Next, we need to show that $x_u \times x_v = 0$,

$$x_u \times x_v = \begin{vmatrix} a\cos u\cos v & b\cos u\sin v & -c\sin u \\ -a\sin u\sin v & b\sin u\cos v & 0 \end{vmatrix}$$

$$= \sin u(bc\sin u\cos v, ac\sin u\sin v, ab\cos u)$$

$$\neq 0$$

Note: $\sin u \neq 0$ since $u \neq 0, \pi$ on the open set $(0, \pi)$.

Finally, the mapping $X : (u, v) \longrightarrow E^3$ is onto.

2. Find the fundamental forms: **I** and **II** of the ellipsoid.

$$x_u = (a\cos u\cos v, b\cos u\sin v, -c\sin u),$$

$$x_v = (-a\sin u\sin v, b\sin u\cos v, 0),$$

$$x_{uu} = (-a\sin u\cos v, -b\sin u\sin v, -c\cos u),$$

$$x_{uv} = (-a\cos u\sin v, b\cos u\cos v, 0),$$

$$x_{vv} = (-a\sin u\cos v, -b\sin u\sin v, 0),$$

$$e_3 = \frac{(bc\sin u\cos v, ac\sin u\sin v, ab\cos u)}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}},$$

$$E = a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u$$

$$= \cos^2 u(a^2\cos^2 v + b^2\sin^2 v) + c^2\sin^2 u,$$

$$F = -a^2\sin u\cos u\sin v\cos v + b^2\sin u\cos u\sin v\cos v + 0$$

$$= (b^2 - a^2)\sin u\cos u\sin v\cos v,$$

$$G = a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v + 0$$

$$= \sin^2 u(a^2\sin^2 v + b^2\cos^2 v),$$

$$L = \frac{-abc}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}},$$

$$M = 0,$$

$$N = \frac{-abc\sin^2 u}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}}.$$

3. Find the normal curvature.

$$\frac{\mathbf{II}}{\mathbf{I}} = \frac{Ldu^2 + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2},$$

where

$$Ldu^2 + Ndv^2 = \frac{-abc}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}}du^2$$
$$+ \frac{-abc\sin^2 u}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}}dv^2$$

and

$$Edu^2 + 2Fdudv + Gdv^2 = (\cos^2 u(a^2\cos^2 v + b^2\sin^2 v) + c^2\sin^2 u)du^2$$
$$+ 2((b^2 - a^2)\sin u\cos u\sin v\cos v)dudv + \sin^2 u(a^2\sin^2 v + b^2\cos^2 v)dv^2.$$

4. Find the principal curvature and directions.

$$k_1, k_2 = \pm\frac{\sqrt{(GL - NE)^2 - 4(LF - ME)(MG - NF)}}{2(LF - ME)}$$
$$= \pm\frac{\sqrt{(GL - NE)^2 - 4(LF)(-NF)}}{2(LF)},$$

where

$$NE = \frac{-abc\sin^2 u\cos^2 u(a^2\cos^2 v + b^2\sin^2 v) + c^2\sin^2 u}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}},$$

$$GL = \frac{-abc\sin^2 u(a^2\sin^2 v + b^2\cos^2 v)}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}},$$

$$NF = \frac{-abc\sin^2 u(b^2 - a^2)\sin u\cos u\sin v\cos v}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}},$$

$$LF = \frac{-abc(b^2 - a^2)\sin u\cos u\sin v\cos v}{\sqrt{c^2\sin^2 u(b^2\cos^2 v + a^2\sin^2 v) + a^2b^2\cos^2 u}}.$$

5. Find the mean curvature.

$$H = \frac{LG + NE}{2(EG - F^2)},$$

where

$$LG + NE = \frac{-abc[\sin^2 u(a^2\sin^2 v + b^2\cos^2 v)]}{\sqrt{(a^2\sin^2 v + b^2\cos^2 v)c^2\sin^2 u + a^2b^2\cos^2 u}}$$
$$+ \frac{-abc\sin^2 u[c^2\sin^2 u + (b^2\sin^2 v + a^2\cos^2 v)cos^2 u]}{\sqrt{(a^2\sin^2 v + b^2\cos^2 v)c^2\sin^2 u + a^2b^2\cos^2 u}}$$

$$= -abc\left[a^2\sin^2 v + b^2\cos^2 v + c^2\sin^2 u + (b^2\sin^2 v + a^2\cos^2 v)\cos^2 u\right]$$

and

$$\begin{aligned} EG - F^2 &= [c^2 \sin^2 u + (b^2 \sin^2 v + a^2 \cos^2 v) \cos^2 u][\sin^2 u(a^2 \sin^2 v + b^2 \cos^2 v)] \\ &\quad -[(b^2 - a^2) \sin u \cos u \sin v \cos v]^2 \\ &= (a^2 c^2 \sin^2 u \sin^2 v + b^2 c^2 \sin^2 u \cos^2 v + a^2 b^2 \cos^2 u)^{\frac{3}{2}}. \end{aligned}$$

6. Find the Gaussian curvature.

$$K = \frac{LN}{EG - F^2},$$

where

$$\begin{aligned} LN &= \frac{(-abc)(-abc \sin^2 u)}{\sqrt{(a^2 \sin^2 v + b^2 \cos^2 v)c^2 \sin^2 u + a^2 b^2 \cos^2 u}^2} \\ &= \frac{a^2 b^2 c^2 \sin^2 u}{(a^2 \sin^2 v + b^2 \cos^2 v)c^2 \sin^2 u + a^2 b^2 \cos^2 u} \end{aligned}$$

and

$$\begin{aligned} EG - F^2 &= [c^2 \sin^2 u + (b^2 \sin^2 v + a^2 \cos^2 v) \cos^2 u][\sin^2 u(a^2 \sin^2 v + b^2 \cos^2 v)] \\ &\quad -[(b^2 - a^2) \sin u \cos u \sin v \cos v]^2 \\ &= a^2 c^2 \sin^4 u \sin^2 v + b^2 c^2 \sin^4 u \cos^2 v + a^2 b^2 \sin^2 u \cos^2 u. \end{aligned}$$

By substituting and simplifying the above into the formula, we obtain:

$$K = \left( \frac{abc}{a^2 c^2 \sin^2 u \sin^2 v + b^2 c^2 \sin^2 u \cos^2 v + a^2 b^2 \cos^2 u} \right)^2.$$

If we look at a point on the ellipsoid we find, $K > 0$. Therefore every point on the ellipsoid is an elliptic point.

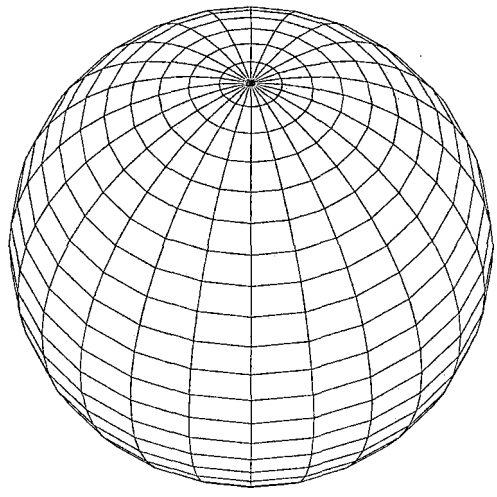

Figure 1.7: The sphere (a).

**Example. THE SPHERE**

In the case where $a = b = c$ of (1.23), we see how the above calculations simplify to the following. The case of the sphere:

$$S = \left\{x^2 + y^2 + z^2 = a^2\right\},$$

where the center of the sphere is at the origin and the radius of the sphere is $a$. We can parameterize S by

$$X(u, v) = (a \sin u \cos v, a \sin u \sin v, a \cos u), \tag{1.24}$$

where $0 < u < \pi$, $0 < v < 2\pi$.

1. Find the fundamental forms: **I** and **II** of the sphere.

$$x_u = (a \cos u \cos v, a \cos u \sin v, -a \sin u),$$

$$x_v = (-a \sin u \sin v, a \sin u \cos v, 0),$$

$$x_{uu} = (-a \sin u \cos v, -a \sin u \sin v, -a \cos u),$$

$$x_{uv} = (-a\cos u\sin v, a\cos u\cos v, 0),$$

$$x_{vv} = (-a\sin u\cos v, -a\sin u\sin v, 0),$$

$$e_3 = -\frac{(a^2\sin u\cos v, a^2\sin u\sin v, a^2\cos u)}{\sqrt{a^2\sin^2 u(a^2\cos^2 v + a^2\sin^2 v) + a^4\cos^2 u}}$$

$$= (\sin u\cos v, \sin u\sin v, \cos u),$$

$$E = \cos^2 u(a^2\cos^2 v + a^2\sin^2 v) + a^2\sin^2 u$$
$$= a^2,$$

$$F = (a^2 - a^2)\sin u\cos u\sin v\cos v$$
$$= 0,$$

$$G = \sin^2 u(a^2\sin^2 u + a^2\cos^2 v)$$
$$= a^2 sin^2 u,$$

$$L = \frac{+a^3}{\sqrt{a^2\sin^2 u(a^2\cos^2 v + a^2\sin^2 v) + a^4\cos^2 u}}$$

$$= +a,$$

$$M = 0,$$

$$N = \frac{+a^3\sin^2 u}{\sqrt{a^2\sin^2 u(a^2\cos^2 v + a^2\sin^2 v) + a^4\cos^2 u}}$$

$$= +a\sin^2 u.$$

2. Find the normal curvature.

$$\begin{aligned}\frac{\mathbf{II}}{\mathbf{I}} &= \frac{Ldu^2 + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \\ &= \frac{+adu^2 + a\sin^2 udv^2}{a^2du^2 + a^2\sin^2 udv^2} \\ &= +\frac{1}{a}.\end{aligned}$$

3. Find the mean curvature.

$$\begin{aligned}H &= \frac{LG + NE}{2(EG - F^2)} \\ &= \frac{(+aa^2\sin^2 u) + (+aa^2\sin^2 u)}{2(a^2a^2\sin^2 u)} \\ &= +\frac{1}{a}.\end{aligned}$$

4. Find the Gaussian curvature.

$$\begin{aligned}K &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{(-a)(-a\sin^2 u)}{(a^2)(a^2\sin^2 u)} \\ &= \frac{1}{a^2}.\end{aligned}$$

5. Find the principal curvature.

$$\begin{aligned}k_1, k_2 &= H \pm \sqrt{H^2 - K} \\ &= +\frac{1}{a}.\end{aligned}$$

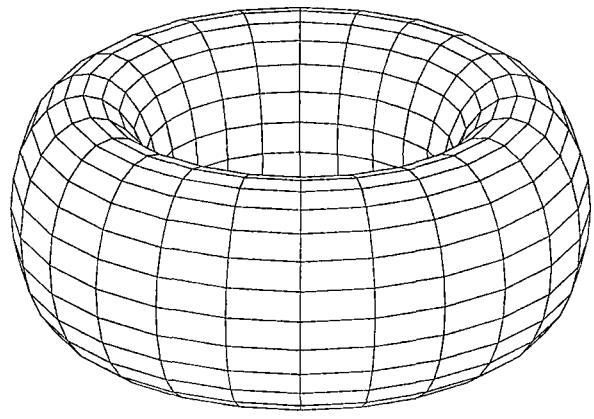

Figure 1.8: The torus (a).

**Example. THE TORUS**

The torus is another compact surface worth noting. In Chapter 3, the Euler characteristic will be found by applying the Gauss-Bonnet theorem to this surface. For now, we will find the first and second fundamental forms, the normal and Gaussian curvature of the torus.

Let the torus be parameterized by:

$$X(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u),$$

where $0 < u < 2\pi$, $0 < v < 2\pi$.

1. Find the fundamental forms: **I** and **II** of the torus.

$$x_u = ((-r\sin u)\cos v, (-r\sin u)\sin v, r\cos u)$$

$$x_v = (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0)$$

$$x_{uu} = (-r\cos u\cos v, -r\cos u\sin v, -r\sin u)$$

$$x_{uv} = (r\sin u\sin v, -r\sin u\cos v, 0)$$

$$x_{vv} = (-(R + r\cos u)\cos v, -(R + r\cos u)\sin v, 0)$$

$$e_3 = (-\cos u \cos v, \sin v \cos u, -\sin u)$$

$$\begin{aligned} E &= r^2 \sin^2 u \cos^2 v + r^2 \sin^2 u \sin^2 v + r^2 \cos^2 u \\ &= r^2 \end{aligned}$$

$$\begin{aligned} F &= (R + r\cos u)(r \sin u) \cos v \sin v - (R + r \cos u)(r \sin u) \cos v \sin v + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} G &= (R + r\cos u)^2 \sin^2 v + (R + r \cos u)^2 \cos^2 v + 0 \\ &= (R + r \cos u)^2 \end{aligned}$$

$$\begin{aligned} L &= r \\ M &= 0 \\ N &= (R + r \cos u) \cos u \end{aligned}$$

2. The normal curvature is

$$\begin{aligned} \frac{\mathbf{II}}{\mathbf{I}} &= \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2} \\ &= \frac{r du^2 + (R + r \cos u) \cos u dv^2}{r^2 du^2 + (R + r \cos u)^2 dv^2}. \end{aligned}$$

3. The Gaussian curvature $K$ is

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{r(R + r \cos u) \cos u}{r^2 (R + r \cos u)^2} \\ &= \frac{\cos u}{r(R + r \cos u)}. \end{aligned}$$

4. The mean curvature $H$ is

$$\begin{aligned} H &= \frac{LG - 2MF + NE}{2(EG - F^2)} \\ &= \frac{r(R + r\cos u)^2 + r^2(R + r\cos u)\cos u}{2r^2(R + r\cos u)^2} \\ &= \frac{R + 2r\cos u}{2r(R + r\cos u)}. \end{aligned}$$

5. The principal curvatures $k_1, k_2$ are

$$\begin{aligned} k_1, k_2 &= H \pm \sqrt{H^2 - K} \\ &= \frac{R + 2r\cos u}{2r(R + r\cos u)} \pm \sqrt{\frac{(R + 2r\cos u)^2}{4r^2(R + r\cos u)^2} - \frac{\cos u}{r(r + R\cos u)}} \\ &= \frac{R + 2r\cos u}{2r(R + r\cos u)} \pm \sqrt{\frac{R^2}{4r^2(R + r\cos u)^2}} \\ &= \frac{R + 2r\cos u}{2r(R + r\cos u)} \pm \frac{R}{2r(R + r\cos u)} \\ &= \frac{1}{r}, \quad \frac{\cos u}{(R + r\cos u)}. \end{aligned}$$

# Chapter 2

# Gauss Bonnet Theorem

**Definition.** Let $S$ be a surface. A region $R \subset S$, the union of a connected open subset of $S$ with its boundary, is said to be **regular** if it is compact and its boundary is a finite union of nonintersecting (simple), closed piecewise regular curves.

**Definition.** A surface $S$ is said to be **orientable** if it can be covered by a family of coordinate neighborhoods such that if a point $\mathbf{p} \in S$ belongs to two neighborhoods of the family, the change of coordinates has a positive Jacobian determinant at $\mathbf{p}$. The choice of such a family is called and **orientation** of $S$. If such a surface is not possible, the surface is said to be **nonorientable**.

A sufficiently small portion of a surface is always orientable. The sphere, torus, and cylinder are orientable surfaces whereas the Mobius strip is a non-orientable surface.

**Theorem 1.** *Let $R \subset S$ be a regular region of an oriented surface $S$ with a boundary $\partial R$ formed by $n$ closed, simple, piecewise regular curves $C_1, \ldots, C_n$. Suppose that each $C_i$ is positively oriented, and let $\phi_1, \ldots, \phi_p$ be the set of all interior angles of the curves $C_1, \ldots, C_n$. Then*

$$\sum_{i=1}^{n} \int_{C_i} \kappa_g(s) ds + \int\int_R K dA = 2\pi\chi(R) - \sum_{i=1}^{p} (\pi - \phi_i),$$

*where $s$ is the arc length, $\kappa_g(s)$ is the geodesic curvature of $C_i$, $K$ is the Gaussian curvature, $dA$ is the element of area of $R$, $\chi(R)$ is the Euler characteristic of the region, and $\phi_i$ are the interior angles of the vertices of the region.*

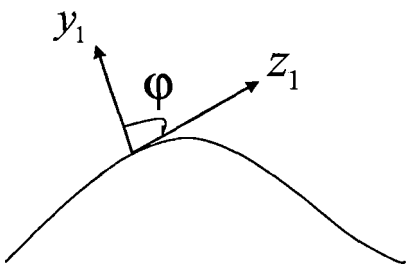


Figure 2.1: The angle between the vectors $y_1$ and $z_1$.

## 2.1 Define the Total Variation

Before we present the proof, we need to consider the total variation of a vector field along a closed curve.

Let $C$ be a closed curve on a surface, $S$ in Euclidean 3-space. Then we have a unit frame $\{e_1, e_2, e_3\}$ at a point $\mathbf{p}$. The vectors $e_1$ and $e_2$ are on the tangent plane at $\mathbf{p}$ and represent the u-curve and v-curve in a small neighborhood of $\mathbf{p}$. The vector $e_3$ is the oriented unit normal vector.

Let $\vec{z_1}(s)$ bet the unit tangent vector field of $C$ parameterized by arc length $s$. We have $\vec{z_1}$ is a linear combination of $e_1$ and $e_2$.

$$\vec{z_1}(s) = a_1(s)e_1 + b_1(s)e_2, \tag{2.1}$$

where $a_1$, $b_1$ are continuously differentiable. Note: $z_1$ is tangent to the surface and assigns each point $\mathbf{p}$ a unit tangent vector of $S$ located on the tangent plane.

Let $\vec{y_1}(s)$ be another unit tangent vector field of $S$ along $C$, where $\nabla\vec{y_1} = 0$ (i.e. $\vec{y_1}$ is a parallel tangent vector field). Likewise, $\vec{y_1}$ is a linear combination of $e_1$ and $e_2$.

$$\vec{y_1}(s) = c_1(s)e_1 + d_1(s)e_2. \tag{2.2}$$

Let $\varphi$ represent the angle between $\vec{y_1}$ and $\vec{z_1}$, see Figure (2.1).

If we look at how this angle changes for each point on $C$, $\dfrac{d\varphi}{ds}$ is independent of the choice of $\vec{y_1}$ and is called the variation of the vector field $\vec{z_1}$ along $C$.

Let $X : U \to S$ be orthogonal parametrization of $S$ where $U \subseteq E^2$ and $X$ keeps orientation of $S$ unchanged such that

$$X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)).$$

Then we have the unit tangent vectors of $u$-curves, $x_u$, with $u$ curve increasing. Let $\theta$ represent the angle between $x_u$ and $\vec{z_1}$. We will assign the tangent vector

$$e_1 = x_u.$$

Orthogonal to the unit tangent vectors, $\vec{y_1}, \vec{z_1}, \vec{e_1}$ of $S$ are the unit tangent vectors $\vec{y_2}, \vec{z_2}, \vec{e_2}$ such that

$$| \vec{y_1}, \vec{y_2}, \vec{e_3} | = | \vec{z_1}, \vec{z_2}, \vec{e_3} | = | \vec{e_1}, \vec{e_2}, \vec{e_3} | = 1,$$

$$\begin{vmatrix} y_{11} & y_{21} & 0 \\ y_{12} & y_{22} & 0 \\ y_{13} & y_{23} & 1 \end{vmatrix} = \begin{vmatrix} z_{11} & z_{21} & 0 \\ z_{12} & z_{22} & 0 \\ z_{13} & z_{23} & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \ .$$

Therefore, $y_{11}y_{22} - y_{21}y_{12} = z_{11}z_{22} - z_{21}z_{12} = 1$. From Equation (2.1), and since $e_1$ is perpendicular to $e_2$ we have:

$$\vec{z_1} = (\cos\theta)\, e_1 + (\sin\theta)\, e_2$$

and

$$\begin{aligned} \vec{z_2} &= \left(\cos\left(\theta + \frac{\pi}{2}\right)\right) e_1 + \left(\sin\left(\theta + \frac{\pi}{2}\right)\right) e_2 \\ &= (-\sin\theta)\, e_1 + (\cos\theta)\, e_2. \end{aligned}$$

And we can conclude that $y_1$ and $y_2$ are as follows:

$$\vec{y_1} = (\cos(\theta - \varphi))\, e_1 + (\sin(\theta - \varphi))\, e_2$$

$$\vec{y_2} = (-\sin(\theta - \varphi))\, e_1 + (\cos(\theta - \varphi))\, e_2.$$

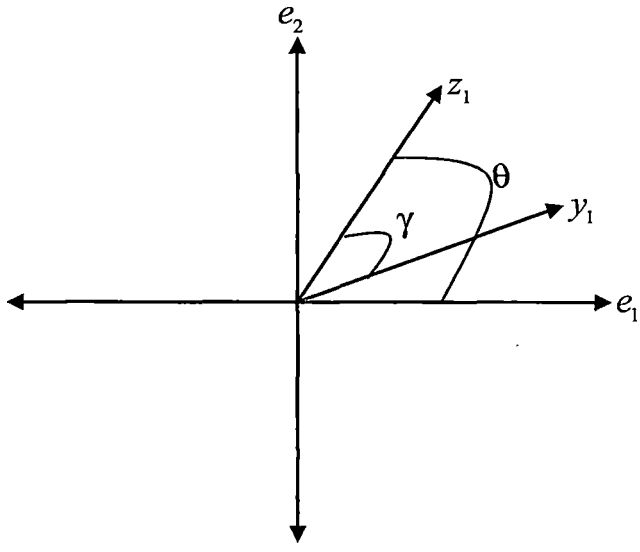


Figure 2.2: The unit tangent vectors $z_1$, $z_2$, $y_1$, and $y_2$.

Since $y_1$ is parallel along $C$ with orthogonal tangent vector $y_2$, both are orthogonal to $\frac{dy_1}{ds}$. Therefore,

$$\frac{dy_1}{ds} \cdot y_1 = \frac{dy_1}{ds} \cdot y_2 = 0,$$

$$\frac{dy_1}{ds} \cdot y_2 = e_2 \cdot \frac{de_1}{ds} + \frac{d(\theta - \varphi)}{ds}.$$

Then,

$$e_2 \cdot \frac{de_1}{ds} + \frac{d(\theta - \varphi)}{ds} = 0,$$

$$e_2 \cdot \frac{de_1}{ds} + \frac{d\theta}{ds} - \frac{d\varphi}{ds} = 0,$$

$$\frac{d\varphi}{ds} = \frac{d\theta}{ds} + e_2 \cdot \frac{de_1}{ds},$$

$$d\varphi = d\theta + e_2 \cdot de_1.$$

Let the variation of the vector field $\vec{z_1}(s)$ be $v = \dfrac{d\varphi}{ds}$ which gives

$$vds = d\theta + e_2 \cdot de_1.$$

In the next section we will show how to integrate the variation of the vector field which will be the result of

$$\int_C vds = \int_C d\theta + \int_C e_2 \cdot de_1. \tag{2.3}$$

## 2.2 Proof of Gauss-Bonnet Theorem

We first prove Gauss-Bonnet Theorem for a simple region $R$ where $\chi(R) = 1$.

**Definition:** Let $S$ be an oriented surface. A region $R \subset S$ is a **simple region** if $R$ is homeomorphic to a disk, and the boundary $\partial R$ of $R$ is the image set of a simple closed piecewise regular parametrized curve $\alpha : I \to S$.

Let $R \subset X(U)$ be a simple region of $S$. Let $C$ be the boundary of $R$. Let $\alpha$ be a curve parametrized by arc length, $s$. Let $\alpha(s_0), \ldots, \alpha(s_k)$ be the vertices of $\alpha$, where $s_i$ represents the distance from $s_0$. Let $\phi_0, \ldots, \phi_k$ be the interior angles at each vertex.

Let $\vec{z_1}(s)$ be the unit tangent vector of $\alpha$ of $R$ (i.e. $\vec{z_1}(s) = X'(s)$). Then the variation, $v$ of $\vec{z_1}$ equals the geodesic curvature at a point $\mathbf{p}$ of $\alpha$ of $R$ of $S$, namely

$$v = \frac{d\varphi}{ds} = k_g.$$

Therefore,

$$\int_C vds = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g ds. \tag{2.4}$$

Let $\theta_i : [s_i, s_{i+1}] \to E^1$ be a differentiable function measuring the angle between $x_u$ and $\alpha'(s)$ at $s \in [s_i, s_{i+1}]$, where $s_{k+1} = s_0$.

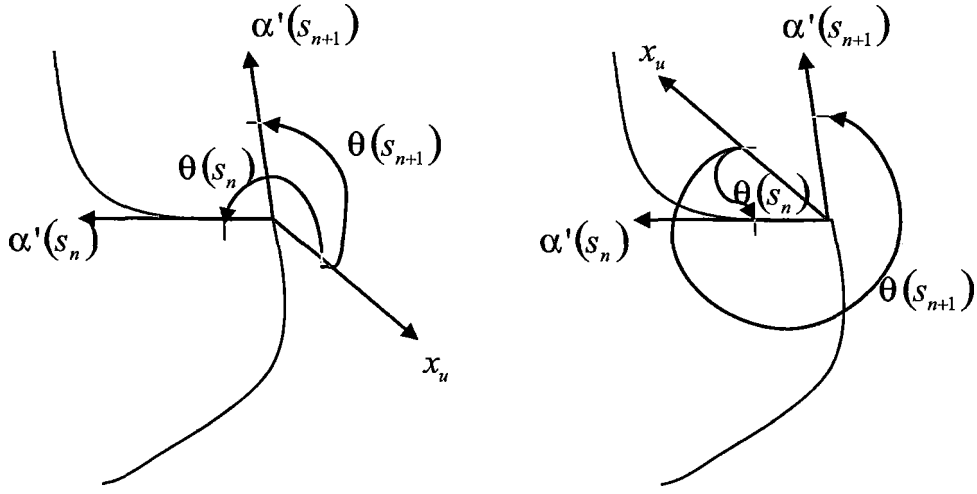


Figure 2.3: The angle measures at a vertex.

By using the rotation index theorem (from reference [Hsi97]), and the figure above we can obtain:

$$\begin{aligned}\int_C d\theta &= \sum_i \int_{C_i} d\theta \\ &= \sum_i [\theta(s_{i+1}) - \theta(s_i)] \\ &= 2\pi - \sum_i (\pi - \phi_i)\,. \end{aligned} \tag{2.5}$$

Since $e_1$ and $e_2$ are $u$,$v$-curves, then $F = M = 0$ and $e_1 = \frac{x_u}{\sqrt{E}}$, $e_2 = \frac{x_v}{\sqrt{G}}$. Then

we have:

$$\begin{aligned} e_2 \cdot de_1 &= \frac{x_v}{\sqrt{G}} \left[ \left( \frac{x_u \frac{E_u}{2\sqrt{E}} - \sqrt{E} x_{uu}}{E} \right) du + \left( \frac{x_v \frac{E_v}{2\sqrt{E}} - \sqrt{E} x_{uv}}{E} \right) dv \right] \\ &= \frac{x_v}{\sqrt{G}} \left[ \left( \frac{x_u E_u}{2\sqrt{E^3}} - \frac{x_{uu}}{\sqrt{E}} \right) du + \left( \frac{x_u E_v}{2\sqrt{E^3}} - \frac{x_{uv}}{\sqrt{E}} \right) dv \right] \\ &= \left( \frac{x_v \cdot x_u E_u}{2E\sqrt{EG}} - \frac{x_v \cdot x_{uu}}{\sqrt{EG}} \right) du + \left( \frac{x_u \cdot x_v E_v}{2E\sqrt{EG}} - \frac{x_v \cdot x_{uv}}{\sqrt{EG}} \right) dv \ . \end{aligned}$$

Since we have

$$E = x_u \cdot x_u, \quad F = x_u \cdot x_v = x_v \cdot x_u = 0, \quad G = x_v \cdot x_v,$$

$$E_v = 2x_u \cdot x_{uv} = -2x_{uu} \cdot x_v \quad \text{and} \quad G_u = 2x_{uv} \cdot x_v,$$

the above simplifies to

$$\begin{aligned} e_2 \cdot de_1 &= \left( 0 - \frac{E_v}{2\sqrt{EG}} \right) du + \left( 0 - \frac{-G_u}{2\sqrt{EG}} \right) dv \\ &= \frac{1}{\sqrt{EG}} \left[ \left( -\frac{E_v}{2} \right) du + \left( \frac{G_u}{2} \right) dv \right] \\ &= \frac{\sqrt{G}}{\sqrt{E}} \left[ \left( -\frac{E_v}{2G} \right) du + \left( \frac{G_u}{2G} \right) dv \right] . \end{aligned}$$

Likewise, since we have $F = 0$ the Christoffel symbols simplify to the following:

$$\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^2_{11} = -\frac{E_v}{2G}, \quad \Gamma^1_{12} = \frac{E_v}{2E}, \tag{2.6}$$

$$\Gamma^2_{12} = \frac{G_u}{2G}, \quad \Gamma^2_{22} = \frac{G_v}{2G}.$$

Then,

$$e_2 \cdot de_1 = \frac{\sqrt{G}}{\sqrt{E}} \left( \Gamma^2_{11} du + \Gamma^2_{12} dv \right) . \tag{2.7}$$

Now we will apply Green's Theorem which states the following:

**Theorem 2.** *Let $R$ be a simply connected region with a piecewise smooth boundary $C$, oriented counterclockwise. If $P$ and $Q$ have continuous partial derivatives in an open region containing $R$, then*

$$\int_C P du + Q dv = \int\int_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du dv. \tag{2.8}$$

Applying Green's Theorem to integrate (2.7), we have:

$$\begin{aligned}\int_C e_2 \cdot de_1 &= \int_C \frac{\sqrt{G}}{\sqrt{E}} \left(\Gamma_{11}^2 du + \Gamma_{12}^2 dv\right) \\ &= \int\int_R \left[\left(\sqrt{\frac{G}{E}}\Gamma_{12}^2\right)_u - \left(\sqrt{\frac{G}{E}}\Gamma_{11}^2\right)_v\right] du dv. \end{aligned} \tag{2.9}$$

In order to simplify this double integral we need to do show some calculations using (1.17), (1.18) and (2.6). Before we find the difference of the partial derivatives, we will look at each alone.

$$\begin{aligned}\left(\sqrt{\frac{G}{E}}\Gamma_{12}^2\right)_u &= \left(\sqrt{\frac{G}{E}}\right)_u \Gamma_{12}^2 + \sqrt{\frac{G}{E}} \left(\Gamma_{12}^2\right)_u \\ &= \frac{1}{2}\left(\frac{G}{E}\right)^{-1/2} \left(\frac{G}{E}\right)_u \Gamma_{12}^2 + \sqrt{\frac{G}{E}} \left(\Gamma_{12}^2\right)_u \\ &= \frac{1}{2}\sqrt{\frac{E}{G}} \left(\frac{EG_u - GE_u}{E^2}\right) \Gamma_{12}^2 + \sqrt{\frac{G}{E}} \left(\Gamma_{12}^2\right)_u \\ &= \frac{1}{2E^2}\sqrt{\frac{E}{G}} \left[E(G_u\Gamma_{12}^2 - G_v\Gamma_{11}^2)\right] + \sqrt{\frac{G}{E}} \left(\Gamma_{12}^2\right)_u \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{EG}}(G_u\Gamma_{12}^2 - G_v\Gamma_{11}^2) + \sqrt{\frac{G}{E}}\left(\Gamma_{12}^2\right)_u \\
&= \frac{G_u}{2\sqrt{EG}}\Gamma_{12}^2 - \frac{G_v}{2\sqrt{EG}}\Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{12}^2\right)_u \\
&= \frac{G}{\sqrt{EG}}\Gamma_{12}^2\Gamma_{12}^2 - \frac{G}{\sqrt{EG}}\Gamma_{22}^2\Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{12}^2\right)_u \\
&= \sqrt{\frac{G}{E}}\Gamma_{12}^2\Gamma_{12}^2 - \sqrt{\frac{G}{E}}\Gamma_{22}^2\Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{12}^2\right)_u \\
&= \sqrt{\frac{G}{E}}\left[\Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{22}^2\Gamma_{11}^2 + \left(\Gamma_{12}^2\right)_u\right] \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
\left(\sqrt{\frac{G}{E}}\Gamma_{11}^2\right)_v &= \left(\sqrt{\frac{G}{E}}\right)_v \Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \frac{1}{2}\left(\frac{G}{E}\right)^{-1/2}\left(\frac{G}{E}\right)_v \Gamma_{12}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \frac{1}{2}\sqrt{\frac{E}{G}}\left(\frac{EG_v - GE_v}{E^2}\right)\Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \frac{1}{2E^2}\sqrt{\frac{E}{G}}G(E_u\Gamma_{12}^2 - E_v\Gamma_{11}^2) + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \frac{\sqrt{EG}}{2E^2}(E_u\Gamma_{12}^2 - E_v\Gamma_{11}^2) + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \frac{E_u\sqrt{EG}}{2E^2}\Gamma_{12}^2 - \frac{E_v\sqrt{EG}}{2E^2}\Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \frac{\sqrt{EG}}{E}\Gamma_{11}^1\Gamma_{12}^2 - \frac{\sqrt{EG}}{E}\Gamma_{12}^1\Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \sqrt{\frac{G}{E}}\Gamma_{11}^1\Gamma_{12}^2 - \sqrt{\frac{G}{E}}\Gamma_{12}^1\Gamma_{11}^2 + \sqrt{\frac{G}{E}}\left(\Gamma_{11}^2\right)_v \\
&= \sqrt{\frac{G}{E}}\left[\Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{12}^1\Gamma_{11}^2 + \left(\Gamma_{11}^2\right)_v\right] \qquad (2.11)
\end{aligned}
$$

Substituting (2.10) and (2.11) into (2.9), we have:

$$\begin{aligned}
\int_C e_2 \cdot de_1 &= \int\int_R \sqrt{\frac{G}{E}}\left[(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v\right] dudv \\
&\quad + \sqrt{\frac{G}{E}}\left[\Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{22}^2\Gamma_{11}^2 - \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{12}^1\Gamma_{11}^2\right] dudv \\
&= \sqrt{\frac{G}{E}}\left[(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{22}^2\Gamma_{11}^2 - \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{12}^1\Gamma_{11}^2\right] dudv \\
&= \int\int_R \sqrt{\frac{G}{E}}\left[-E\frac{LN - M^2}{EG - F^2}\right] dudv \\
&= \int\int_R \sqrt{\frac{G}{E}}\left[-E\frac{LN}{EG}\right] dudv \\
&= \int\int_R \frac{\sqrt{EG}}{E}\left[-EK\right] dudv \\
&= \int\int_R -K\sqrt{EG} dudv \\
&= -\int\int_R K dA, \qquad (2.12)
\end{aligned}$$

where $dA = \sqrt{EG}dudv$.

By substituting (2.4), (2.5), and (2.12) into (2.3), we have obtained the Gauss-Bonnet formula for a simple region, $R$, lying in a coordinate neighborhood of a surface,

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g ds + \int\int_R K dA = 2\pi - \sum_{i=0}^{k}(\pi - \phi_i). \qquad (2.13)$$

Now we will consider the regular region $R$ where we have a triangulation $\{\mathcal{T}_j\}$ of $R$ such that every triangle $\mathcal{T}_j$, is contained in a coordinate neighborhood of a family of orthogonal parametrizations. By applying the Gauss-Bonnet formula (2.13) to every triangle of $R$, we have

$$\begin{aligned}
\mathcal{T}_1 \quad : \quad & \int_{s_1^1}^{s_2^1} k_g ds + \int_{s_2^1}^{s_3^1} k_g ds + \int_{s_3^1}^{s_1^1} k_g ds + \int\int_{\mathcal{T}_1} K dA \\
= \quad & 2\pi - \left(\pi - \alpha_1^1 + \pi - \alpha_2^1 + \pi - \alpha_3^1\right) \\
= \quad & \alpha_1^1 + \alpha_2^1 + \alpha_3^1 - \pi, \qquad (2.14) \\
\mathcal{T}_2 \quad : \quad & \int_{s_1^2}^{s_2^2} k_g ds + \int_{s_2^2}^{s_3^2} k_g ds + \int_{s_1^2}^{s_3^2} k_g ds + \int\int_{\mathcal{T}_2} K dA \\
= \quad & 2\pi - \left(\pi - \alpha_1^2 + \pi - \alpha_2^2 + \pi - \alpha_3^2\right) \\
= \quad & \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \pi, \qquad (2.15) \\
\vdots \quad = \quad & \vdots, \qquad (2.16) \\
\mathcal{T}_k \quad : \quad & \int_{s_1^k}^{s_2^k} k_g ds + \int_{s_2^k}^{s_3^k} k_g ds + \int_{s_3^k}^{s_1^k} k_g ds + \int\int_{\mathcal{T}_k} K dA \\
= \quad & 2\pi - \left(\pi - \alpha_1^k + \pi - \alpha_2^k + \pi - \alpha_3^k\right) \\
= \quad & \alpha_1^k + \alpha_2^k + \alpha_3^k - \pi, \qquad (2.17)
\end{aligned}$$

where $k$ is the number of triangles and $\alpha_i^j$ for $i = 1, 2, 3$, $j = 1, ..., k$ are the interior angles of each triangle.

The integrals of geodesics curvature along each interior edge cancel each other, since opposite orientations are induced on the common edge of every pair of adjacent triangles, namely

$$\int_{s_i}^{s_{i+1}} k_g ds + \int_{s_{i+1}}^{s_i} k_g ds = 0$$

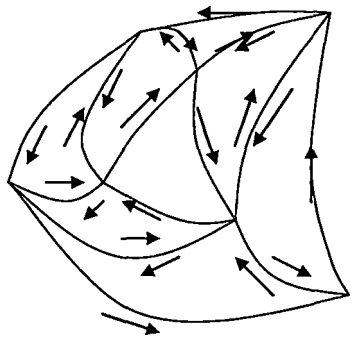

Figure 2.4: An example of a triangulation.

and

$$\int\int_{\mathcal{T}_1} KdA + \int\int_{\mathcal{T}_2} KdA + \ldots + \int\int_{\mathcal{T}_k} KdA = \int\int_R KdA.$$

When we add up the equations (2.14), (2.15), (2.16) and (2.17) we obtain:

$$\sum_{i=1}^{n} \int_{C_i} k_g ds + \int\int_R KdA \;=\; \sum_{i=1}^{k} (\alpha_1^i + \alpha_2^i + \alpha_3^i - \pi). \tag{2.18}$$

Now we look at the relation between the number of angles, the number of exterior and interior vertices and the number of exterior and interior edges.

**Proposition.** For a triangulation $\mathcal{T}$, the sum of the number of exterior vertices and the number of angles is equal to twice the number of edges, namely

$$l + 3k = 2e, \tag{2.19}$$

where $l$ is the number of exterior vertices, $e$ is the number of edges and $k$ is the number of triangles.

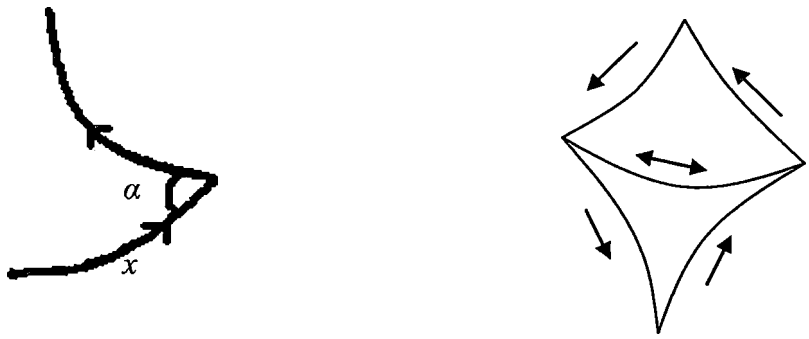


Figure 2.5: The edge associated with its angle.

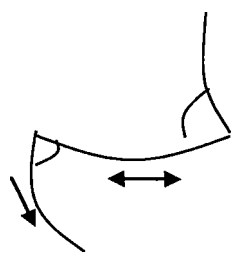

Figure 2.6: The two angles associated with one edge.

**Proof.** We associate each angle, $\alpha$ with its initial edge, $x$, see Figure (2.5). But for each interior edge, there are two angles associated to it, see Figure (2.6).

For each exterior edge there is one angle and one exterior vertex associated to it, see Figure (2.5). Therefore, the sum of number of exterior vertices and the number of angle is equal to twice the number of edges $= 2e$.

Now, we can finish looking at (2.18) applied to a triangulation $\mathcal{T} = \{T_i\}_{i=1}^{k}$ of $R$.

Let $f$ be the number of faces (i.e. the number of triangles in the triangulation). Let $e$ be the number of edges. Let $v$ be the number of vertices. Then the number $v - e + f = \chi$ is the Euler characteristic of the triangulation. Since $k$ is the number of triangles, it is also the number of faces, $f$. (i.e. $k = f$) Let $m$ be the number of interior vertices. Let $l$ be the number of exterior vertices. Let $\phi_j$ be the interior angle at each exterior vertex. Finally, we obtain:

$$\begin{aligned}
\sum_{i=1}^{n}\int_{C_i}\kappa_g ds + \int\int_R K dA &= \sum_{i=1}^{k}[(\alpha_1^i + \alpha_2^i + \alpha_3^i) - \pi] \\
&= \sum_{i=1}^{k}(\alpha_1^i + \alpha_2^i + \alpha_3^i) - k\pi \\
&= 2m\pi + \sum_{j=1}^{p}\phi_j + (l-p)\pi - k\pi \\
&= 2m\pi - \sum_{j=1}^{p}(\pi - \phi_j) + l\pi - k\pi \\
&= 2m\pi + l\pi - k\pi - \sum_{j=1}^{p}(\pi - \phi_j) \\
&= 2m\pi + l\pi - k\pi - \sum_{j=1}^{p}(\pi - \phi_j) \\
&= 2v\pi - l\pi - 3k\pi + 2k\pi - \sum_{j=1}^{p}(\pi - \phi_j) \\
&= 2v\pi - (l+3k)\pi + 2k\pi - \sum_{j=1}^{p}(\pi - \phi_j)
\end{aligned}$$

$$
\begin{aligned}
&= 2v\pi - (2e)\pi + 2f\pi - \sum_{j=1}^{p}(\pi - \phi_j) \\
&= 2\pi(v - e + f) - \sum_{j=1}^{p}(\pi - \phi_j) \\
&= 2\pi\chi - \sum_{j=1}^{p}(\pi - \phi_j).
\end{aligned}
$$

In conclusion, we have the Gauss Bonnet Theorem:

$$\sum_{i=1}^{n}\int_{C_i}\kappa_g(s)ds + \int\int_R KdA = 2\pi\chi(R) - \sum_{i=1}^{p}(\pi - \phi_i).$$

■

# Chapter 3

# Applications

Now that we have stated and proved the Gauss-Bonnet theorem we can make application of it to some special surfaces. We can use it to find the Euler characteristic of a surface or we can show what the area of a surface is. Here are some examples of some geometries and their areas.

**Corollary.** For a compact surface,

$$\int\int_S KdA = 2\pi\chi(S). \tag{3.1}$$

We can use the Gauss-Bonnet theorem to either find the area the compact surface given its Euler characteristic or vice versa.

**Example.** If we look at the sphere parameterized by

$$X(u,v) = (a\sin u\cos v, a\sin u\sin v, a\cos u),$$

of radius $a$ and where $0 < u < \pi$, $0 < v < 2\pi$, we have already calculated the first and second fundamental forms and the different types of curvatures.

Recall the following calculations from Chapter 1:

$$\begin{aligned} E &= a^2, \\ F &= 0, \\ G &= a^2\sin^2 u, \end{aligned}$$

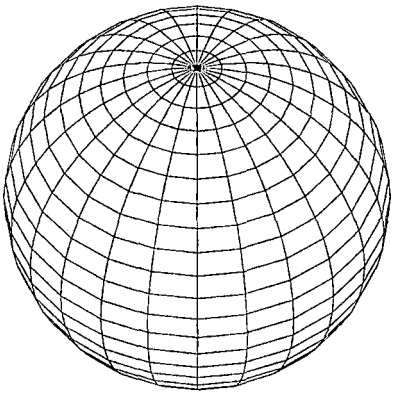

Figure 3.1: The sphere (b).

$$\begin{aligned} L &= +a, \\ M &= 0, \\ N &= +a^2 \sin^2 u. \end{aligned}$$

The normal curvature is:

$$\begin{aligned} \frac{\mathbf{II}}{\mathbf{I}} &= \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \\ &= +\frac{1}{a}. \end{aligned}$$

The Gaussian curvature is:

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{1}{a^2}. \end{aligned}$$

Therefore we can calculate the euler characteristic since $2\pi\chi(S)$ is equal to the following:

$$\begin{aligned} \int\int_S KdA &= \int\int_S \frac{a^2 \sin u}{a^2} dudv \\ &= \int\int_S \sin u dudv \end{aligned}$$

$$= \int -\cos u \Big|_0^{\pi} dv = \int -(-1-1)dv$$

$$= \int 2dv = 2v|_0^{2\pi}$$

$$= 4\pi,$$

where

$$dA = \sqrt{EG - F^2} = a^2 \sin u du dv.$$

In conclusion, $\chi = 2$ from:

$$\int \int_S K dA = 2\pi\chi(S) = 4\pi.$$

**Example.** We can also use the Gauss-Bonnet theorem of compact surfaces to find the Euler characteristic of the torus parameterized by:

$$X(u, v) = ((R + r\cos u)\cos v), (R + r\cos u)\sin v, r\sin u),$$

where $0 < u < 2\pi$, $0 < v < 2\pi$.

From Chapter 1, we have already calculated the first and second fundamental forms and the different types of curvatures. Recall the following calculations:

$$\begin{aligned} E &= r^2, \\ F &= 0, \\ G &= (R + r\cos u)^2, \\ L &= r, \\ M &= 0, \\ N &= (R + r\cos u)\cos u. \end{aligned}$$

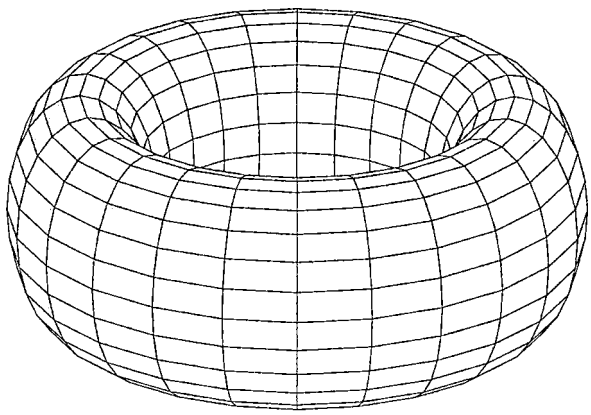

Figure 3.2: The torus (b).

The normal curvature is:

$$\begin{aligned}\frac{\mathbf{II}}{\mathbf{I}} &= \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \\ &= \frac{rdu^2 + (R + r\cos u)\cos u dv^2}{r^2du^2 + (R + r\cos u)^2dv^2}.\end{aligned}$$

The Gaussian curvature is:

$$\begin{aligned}K &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{\cos u}{r(r + R\cos u)}.\end{aligned}$$

By substituting the above into (3.1), we have:

$$\begin{aligned}\int\int_S KdA &= \int\int_S \frac{\cos u}{r(r + R\cos u)} r(r + R\cos u)dudv \\ &= \int\int_S \cos u dudv \\ &= \int -\sin u\Big|_0^{2\pi} dv = \int -(0 - 0)dv\end{aligned}$$

$$= \int 0dv = 0|_0^{2\pi}$$

$$= 0.$$

Therefore $\chi = 0$ from:

$$\int\int_S KdA = 2\pi\chi(S) = 0.$$

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